On isometries of spectral triples associated to AF-algebras and crossed products

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Abstract

We examine the structure of two possible candidates of isometry groups for the spectral triples on AF-algebras introduced by Christensen and Ivan. In particular, we completely determine the isometry group introduced by Park, and observe that these groups coincide in the case of the Cantor set. We also show that the construction of spectral triples on crossed products given by Hawkins, Skalski, White and Zacharias, is suitable for the purpose of lifting isometries.

1 Introduction

The reconstruction theorem by Connes is a smooth version of the Gelfand duality; it shows that both the geodesic distance and the volume form on a compact Riemannian space are encoded in a particular spectral triple ([5]). The non-commutative differential geometry paradigm is that spectral triples on C^*-algebras should allow the investigation of geometric properties in the non-abelian setting.

Many authors studied the nature of such features in different specific cases; this lead for example to the notion of compact quantum metric space, as defined by Rieffel ([17]). Examples of non-commutative manifolds have been studied in the case of the reduced C^*-algebra of a discrete group ([5]), AF-algebras ([4]), crossed products ([10]), and many other instances.

From the point of view of non-commutative geometry, a natural problem is to understand what is the right notion of isometry for a non-commutative manifold. Such topic was investigated in [14, 15, 16, 13, 6] and at the time being the two definitions of non-commutative isometry appearing in these manuscripts are the only natural candidates known to the authors. More precisely, these are the automorphisms implemented by unitaries commuting with the Dirac operator and the automorphisms leading to preservation of the Connes distance on the state space, giving rise to the two groups Iso and ISO, respectively.

In the present work the authors investigate properties of the Iso- and ISO-group in some specific cases, namely the spectral triples constructed by Christensen and Ivan on AF-algebras and the spectral triples on the crossed product of a C^*-algebra with a discrete group, as defined in [10]. A concrete description of the Iso-group is provided in the case of the spectral triples for AF-algebras (under a natural nondegeneracy condition), namely

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its elements are the automorphisms which preserve the filtration and the given reference faithful state. In the particular case of some natural spectral triple on the Cantor set, this group actually coincides with the ISO-group. Moreover, it is proved that in the case of UHF-algebras of type n^∞, if the eigenvalues of the Dirac operator grow fast enough, the ISO-group cannot contain all the switches of the tensor factors. In the case of the CAR-algebra, we explicitly compute the Connes distance between certain states; as a consequence, none of the switch automorphisms can appear in the ISO-group at all. In Section 4 it is shown a procedure to lift the elements of the Iso-group of a spectral triple for a C^*-algebra endowed with an action of a countable discrete group to the Iso-group of the spectral triple of the crossed product given in [10].

2 Preliminaries

2.1 The definition of a spectral triple

The geometric properties of a compact Riemannian spin manifold can be reconstructed from the algebraic data contained in the way the Dirac operator and the measurable bounded functions interact when acting on the Hilbert space of L^2-spinors ([5]). These data are encoded in the notion of spectral triple, which we recall in the setting of arbitrary unital C^*-algebras.

Definition 2.1. Let A be a unital C^*-algebra represented on a Hilbert space H and D be an unbounded self-adjoint operator on H. Denote $A_D = \{a \in A \mid a \operatorname{dom}(D) \subset \operatorname{dom}(D)\}$. We say that (A, H, D) is a *spectral triple* if the following conditions are satisfied:

(i) the set $\{a \in A_D \mid [D, a]$ extends to a bounded operator on $H\}$ is norm-dense in A;

(ii) The operator $(1 + D^2)^{-1}$ is compact.

In this case, the operator D is a *Dirac operator*.

As observed by Connes, given a spectral triple (A, H, D), it is possible to define a pseudo-metric d_D on $\mathcal{S}(A)$, the state space of A, by the formula

$$d_D(\phi, \psi) = \sup_{a \in A \,:\, \|[D,a]\| \leqslant 1} |\phi(a) - \psi(a)|.$$

In the case where the spectral triple is the natural one associated to a compact Riemannian spin manifold, this formula gives back the geodesic distance ([5], chapt. 6, par. 1); there are also examples of non-commutative C^*-algebras admitting spectral triples for which d_D is a metric on the state space inducing the w^*-topology (c.f. [17]).

2.2 Spectral triples on AF-algebras

We briefly recall the construction of spectral triples on AF-algebras given in [4]. Let $A = \lim_n A_n$ be a unital AF-algebra associated to a filtration of finite-dimensional C^*-algebras A_n and assume $A_0 = \mathbb{C}1$. Let ϕ be a faithful state on A and H_ϕ the associated

GNS-Hilbert space, ξ the corresponding cyclic vector. For every $n \in \mathbb{N}$ let $H_n = A_n \xi$ and $P_n : H_\phi \to H_n$ be the associated orthogonal projection; we define $K_n = H_n \ominus H_{n-1}$, $Q_n = P_n - P_{n-1}$. Given a sequence of positive real numbers $\lambda_n \to \infty$, with $\lambda_0 = 0$, we let $D_{\{\lambda_n\}}$ be the unbounded self-adjoint operator on H_ϕ given by $D = \sum_n \lambda_n Q_n$ with its natural domain; this is a Dirac operator for a spectral triple $(A, H_\phi, D_{\{\lambda_n\}})$ on A. The elements of the dense subalgebra $\bigcup_n A_n \subset A_{D_{\{\lambda_n\}}}$ have bounded commutators with this Dirac operator. Following [10], spectral triples constructed in this way will be referred to as Christensen-Ivan spectral triples and we will drop the suffix $\{\lambda_n\}$ appearing in the Dirac operator where no confusion is likely to arise. As shown in [4] Theorem 2.1, if the sequence of eigenvalues $\{\lambda_n\}$ grows rapidly enough, the pseudo-metric induced by $D_{\{\lambda_n\}}$ on the state space of A is actually a metric and it induces the w^*-topology. We will see an application of this fact in Lemma 3.6.

2.3 Isometries of non-commutative spaces

Park introduced in [14] the concept of isometry in the non-commutative setting (see also [13]). He showed that in the case of the spectral triple given by the continuous functions on a compact Riemannian oriented manifold, the Hilbert space of complex L^2-forms and the de Rham operator as Dirac, this concept coincides with the ordinary notion of isometry. He also studied the group of such isometries in some non-commutative cases.

Definition 2.2 ([14]). Let (A, H, D) be a spectral triple. An element $\alpha \in \mathrm{Aut}(A)$ belongs to the group $\mathrm{Iso}(A, H, D)$ if α is implemented on H by a unitary operator which leaves the domain of the Dirac operator D invariant and commutes with D.

In view of the possibility to characterize the distance between two points in a compact Riemannian spin manifold M in terms of the metric d_D on the state space of $C(M)$, another possible non-commutative generalization of isometry which is worth studying is given by:

Definition 2.3 ([6]). Let (A, H, D) be a spectral triple. An element $\alpha \in \mathrm{Aut}(A)$ belongs to the group $\mathrm{ISO}(A, H, D)$ if $d_D(\phi \circ \alpha, \psi \circ \alpha) = d_D(\phi, \psi)$ for every $\phi, \psi \in \mathcal{S}(A)$.

While it is always the case that $\mathrm{Iso}(A, H, D) \subset \mathrm{ISO}(A, H, D)$, there are simple examples in which these two groups do not coincide.

Proposition 2.4. *Let* $A = M_2(\mathbb{C})$ *and let* D *be a diagonal self-adjoint element of* A *with two distinct eigenvalues. Then* $\mathrm{Iso}(A, \mathbb{C}^2, D) \neq \mathrm{ISO}(A, \mathbb{C}^2, D)$.

Proof. The "flip" unitary $U = (1 - \delta_{i,j})_{i,j=1}^2$ does not commute with D, from which it follows that $\mathrm{ad}(U) \notin \mathrm{Iso}(A, \mathbb{C}^2, D)$; we will see that U implements an element of $\mathrm{ISO}(A, \mathbb{C}^2, D)$. For let $\phi, \psi \in \mathcal{S}(A)$ and note that

$$d_D(\phi \circ \mathrm{ad}(U), \psi \circ \mathrm{ad}(U)) = \sup_{a \in A: \|[D,a]\| \leqslant 1} |\phi(UaU^*) - \psi(UaU^*)|$$

$$= \sup_{a \in A: \|[D,U^*aU]\| \leqslant 1} |\phi(a) - \psi(a)|.$$

Hence it is enough to show that $\|[D, UaU^*]\| = \|[D, a]\|$ for every $a \in A$. This follows from the equality $[D, U^*aU] = -U^*[D, a]U$ for every $a \in A$. $\qquad \square$

Remark 2.5. The fact that the automorphism induced by the flip unitary of Proposition 2.4 preserves the Connes distance between *pure* states already follows from the computation in [11] Proposition 2.

Other comparison results concerning the two isometry groups for the case of Cuntz algebras have been obtained in [6].

3   Isometries of Christensen-Ivan spectral triples

This section contains information about the isometry groups associated to spectral triples on AF-algebras. We completely characterize the Iso-groups for the cases at hand and collect some results concerning the "size" of the ISO-group.

Let $A = \overline{\bigcup_n A_n}$ be an AF algebra, ϕ a faithful state on A and $(\pi_\phi, H_\phi, \xi_\phi)$ the associated GNS triple. From now on we identify A with its image under π_ϕ and write ξ for the cyclic and separating vector ξ_ϕ. Let (A, H_ϕ, D) be the Christensen-Ivan spectral triple associated to the given generating family of finite-dimensional subalgebras A_n and the diverging sequence of non-negative real numbers $\{\lambda_n\}$ (here, $A_0 = \mathbb{C}I$ and $\lambda_0 = 0$).

Theorem 3.1. *Let (A, H_ϕ, D) be as above and suppose that the sequence $\{\lambda_n\}$ satisfies $\lambda_i \neq \lambda_j$ for every $i \neq j$. Then*

$$\mathrm{Iso}(A, H_\phi, D) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha(A_i) = A_i \ \forall i, \ \phi \circ \alpha = \phi\}$$

In particular, if $A = \otimes_{i=0}^\infty M_{n_i}$, with $M_{n_0} = \mathbb{C}$, is a UHF-algebra, then

$$\mathrm{Iso}(A, H_\phi, D) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha = \otimes_{i=1}^\infty \alpha_i, \ \alpha_i \in \mathrm{Aut}(M_{n_i}), \ \phi \circ \alpha = \phi\}$$

and if $\phi = \otimes_i \phi_i$ is a product state, then

$$\mathrm{Iso}(A, H_\phi, D) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha = \otimes_{i=1}^\infty \alpha_i, \ \alpha_i \in \mathrm{Aut}(M_{n_i}), \ \phi_i \circ \alpha_i = \phi_i \ \text{for every } i \in \mathbb{N}\}$$

$$\simeq \prod_{i=1}^\infty \{u_i \in U_{n_i} \mid \phi_i \circ \mathrm{ad} u_i = \phi_i \ \text{for every } i \in \mathbb{N}\}/S^1.$$

Proof. Let $\alpha \in \mathrm{Iso}(A, H_\phi, D)$ and $U \in U(H_\phi)$ be such that $\alpha = \mathrm{ad}(U)$. Since the coefficients λ_n are pairwise different, the condition that $[D, U] = 0$ is equivalent, for $U \in U(H)$, to the condition that $U H_i \subset H_i$ for every $i \in \mathbb{N}$, where H_i is the finite-dimensional Hilbert space $A_i \xi$, and thus $U H_i = H_i$, for every i. In particular, $U\xi \in \mathbb{C}\xi$ and, without loss of generality, we may even assume that $U\xi = \xi$. Anyway, it readily follows that $\phi \circ \alpha = \phi$. Moreover, note that, if $U \in B(H)$ leaves every H_i invariant, then for every $i \in \mathbb{N}$ and $a \in A_i$ we have $U a U^* \xi \in H_i$. Since ξ is separating it follows that $U a U^* \in A_i$; hence $\mathrm{ad}(U)$ respects the specified filtration of A. Hence $\mathrm{Iso}(A, H_\phi, D) \subset \{\alpha \in \mathrm{Aut}(A) \mid \alpha(A_i) = A_i \ \forall i, \ \phi \circ \alpha = \phi\}$. Let now α be an automorphism of A which leaves every A_i invariant and preserves the faithful state ϕ under precomposition. Then the linear map $a\xi \to \alpha(a)\xi$ defines a unitary operator on H; hence α is unitarily implemented and since the unitaries realizing α leave every H_i invariant, they commute with D.

Suppose now $A = \otimes_{i=0}^{\infty} M_{n_i}$ be a UHF-algebra and let $\alpha \in \mathrm{Iso}(A, H_\phi, D)$ be implemented by $U \in B(H_\phi)$. Since for every $i \in \mathbb{N}$ $\mathrm{ad}(U)$ leaves both M_{n_i} and $M_{n_i} \otimes M_{n_{i+1}}$ invariant, it also leaves $M_{n_{i+1}}$ invariant, being the commutant of M_{n_i} in $M_{n_i} \otimes M_{n_{i+1}}$. Hence every $\alpha \in \mathrm{Iso}(A, H_\phi, D)$ decomposes as $\alpha = \otimes_{n=1}^{\infty} \alpha_i$, where $\alpha_i \in \mathrm{Aut}(M_{n_i})$. On the other hand, every automorphism of the form $\alpha = \otimes_{i=1}^{\infty} \alpha_i$ with $\alpha_i \in \mathrm{Aut}(M_{n_i})$ which satisfies $\phi \circ \alpha = \phi$ belongs to $\mathrm{Iso}(A, H_\phi, D)$. Now the last statement easily follows. $\qquad\square$

The above argument can be used to obtain similar results in the more general case of spectral triples associated to non-decreasing sequences of real numbers.

Corollary 3.2. *Let $A = \lim_i A_i$ be an AF-algebra, ϕ a faithful state on A and λ_i be a monotone non-decreasing sequence associated to a Dirac operator on H_ϕ implementing a Christensen-Ivan spectral triple. For every $i \in \mathbb{N}$ let $n_i := \max\{k \geqslant i \mid \lambda_k = \lambda_i\}$ and $k_i := \min\{k \leqslant i \mid \lambda_k = \lambda_i\}$. Then*

$$\mathrm{Iso}(A, H_\phi, D) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha(A_{n_i}) = A_{n_i} \; \forall i \in \mathbb{N}, \; \phi \circ \alpha = \phi\}.$$

If $A = \otimes_{i=0}^{\infty} M_{n_i}$ is UHF, then

$$\mathrm{Iso}(A, H_\phi, D) = \{\alpha \in \mathrm{Aut}(A) \mid \alpha = \otimes_{\lambda_i} \alpha_{\lambda_i}, \; \alpha_{\lambda_i} \in \mathrm{Aut}(\otimes_{j=k_i}^{n_i} M_j), \; \phi \circ \alpha_{\lambda_i} = \phi\}$$

Proof. The Dirac operator reads $D = \sum_{\lambda_i \in \{\lambda_i\}_{i \in \mathbb{N}}} \lambda_i (P_{n_i} - P_{k_i})$, where for $j \in \mathbb{N}$, $P_j = P_{H_j}$. Hence $\mathrm{Iso}(A, H_\phi, D)$ can be computed using Theorem 3.1 applied to the AF-structure of A given by the filtration $A = \lim_i A_{n_i}$. The second statement follows from the second part of the proof of Theorem 3.1. $\qquad\square$

Remark 3.3. In the particular case of a UHF-algebra $A = \otimes_{i=0}^{\infty} M_{n_i}$ represented on the GNS-Hilbert space associated to the unique trace τ, Corollary 3.2 gives, for a monotone non-decreasing sequence $\{\lambda_i\}$,

$$\mathrm{Iso}(A, H_\tau, D_{\{\lambda_i\}}) = \prod_{\lambda_i} U_{\prod_{\{j \, : \, \lambda_j = \lambda_i\}} n_j} / S^1$$

Remark 3.4. It is unlikely that $\mathrm{Iso}(A, H_\phi, D)$ be normal in $\mathrm{Aut}(A)$. It is certainly not for $A = \otimes_{i=0}^{\infty} M_{n_i}$ with $n_i = m^k$ for some $2 \leqslant \min\{m, k\}$, for all $i \in \mathbb{N}$ and $\phi = \mathrm{tr}$. In fact, suppose $A = \otimes_{i=0}^{\infty} M_{2^k}$ and consider $\alpha = \mathrm{id}^{\otimes j} \otimes \hat{\alpha} \otimes \mathrm{id}^{\infty}$ for some $j \geqslant 1$, where $\hat{\alpha}(a_1 \otimes a_2 \otimes \ldots \otimes a_k) = a_2 \otimes a_1 \otimes a_3 \otimes \ldots \otimes a_k$. Then $\beta \circ \alpha \circ \beta^{-1} \notin \mathrm{Iso}(A, H_\phi, D)$ for $\beta = \mathrm{id}^{\otimes(j-1)} \otimes \hat{\beta} \otimes \mathrm{id}^{\otimes \infty}$, with $\hat{\beta}((a_1 \otimes \ldots \otimes a_k) \otimes (b_1 \otimes \ldots \otimes b_k)) = (a_1 \otimes \ldots \otimes a_{k-1} \otimes b_1) \otimes (a_k \otimes b_2 \otimes \ldots \otimes b_k)$.

It was observed in [4] that in the case of Christensen-Ivan spectral triples associated to rapidly diverging sequences of real numbers, all the information about the w^*-topology on the state space of the AF-algebra is carried by the Dirac operator, namely the topology induced by the metric associated to D (see the discussion after Definition 2.1) on the state space of A coincides with the w^*-topology ([4] Theorem 2.1 (i)). The following is an application of this fact.

Proposition 3.5. *Let A be a UHF-algebra, tr the trace on A and $\{\lambda_n\}_{n \in \mathbb{N}}$ a diverging sequence of positive real numbers such that the topology induced by the metric associated to $D_{\{\lambda_n\}}$ on $\mathcal{S}(A)$ coincides with the w^*-topology. Let $(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}})$ be the Christensen-Ivan spectral triple associated to this data. Then*

$$\mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}}) \subsetneq \mathrm{Aut}(A)$$

Moreover, $\mathrm{Aut}(A) \backslash \mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}})$ *contains* $\mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}}) \times \mathbb{N}$ *as a subset.*

Proof. In virtue of [4] Theorem 2.1 (i) the pseudometric $d_{D_{\{\lambda_n\}}}$ induced by $D_{\{\lambda_n\}}$ on the state space of A is actually a metric and it induces the w^*-topology. By [18] Corollary 3.8 the action of $\mathrm{Aut}(A)$ on the pure states of A is transitive and by [9] Theorem 2.8 the set of pure states is w^*-dense in the set of states. Hence for every pure state ψ there is a sequence $\{\beta_n\} \subset \mathrm{Aut}(A)$ such that $d_{D_{\{\lambda_n\}}}(\psi \circ \beta_n, \mathrm{tr}) \to 0$. In particular, there is a sequence of positive real numbers $r_n \to 0$, $r_n \neq r_m$ for $n \neq m$ such that the sets $\Omega_n := \{\phi \in \mathcal{S}(A) \mid d_{D_{\{\lambda_n\}}}(\mathrm{tr}, \phi) = r_n\} \neq \varnothing$ for every $n \in \mathbb{N}$ and from the above, given $n \neq m \in \mathbb{N}$, there is always an element $\alpha_{n,m} \in \mathrm{Aut}(A)$ such that $\alpha_{n,m}(\Omega_n) \cap \Omega_m \neq \varnothing$. We claim that given $n, m \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $\alpha_{n,m} \neq \alpha_{n,h} \circ \beta$ for every $h \geqslant l$, $\beta \in \mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}})$; indeed, if this is not the case, then it is possible to find a sequence $\phi_k \in \Omega_n$ such that $\alpha_{n,m}(\phi_k) \to \mathrm{tr}$, by compactness we can suppose ϕ_k converges to a state ϕ, which satisfies $\alpha_{n,m}(\phi) = \mathrm{tr}$, which is impossible. It follows that for every $n \in \mathbb{N}$ there is a sequence m_i such that the elements $\alpha_{n,m_i} \circ \beta_1 \neq \alpha_{n,m_j} \circ \beta_2$ for every $(i, \beta_1) \neq (j, \beta_2) \in \mathbb{N} \times \mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\lambda_n\}})$. $\square$

Let $n \in \mathbb{N}$ and $A = \otimes^\infty M_n$ be a UHF-algebra. For $i, j \in \mathbb{N}$ let $\sigma_{i,j}$ be the automorphism of A given by the permutation of the i-th and j-th tensor components. It follows from Theorem 3.1 that for every $i \neq j$, $\sigma_{i,j}$ does not belong to the Iso-group of a Christensen-Ivan spectral triple for A with pairwise different eigenvalues.

Lemma 3.6. *Let $A = \otimes_i M_{n_i}$ be a UHF-algebra with $n_i = n_j = n$ for every i, j and consider the setting as in the statement of Proposition 3.5. There exists $i \in \mathbb{N}$ such that $\sigma_{i,i+1} \notin \mathrm{ISO}(A, H_{\mathrm{tr}}, D_{\{\alpha_n\}})$.*

Proof. Let $a \in M_n$ be such that $\mathrm{tr}(a^*a) = 1$, ω_a the associated (vector) state, namely $\omega_a(x) = \mathrm{tr}((a^* \otimes 1)x(a \otimes 1))$ and suppose $\omega_a \neq \mathrm{tr}$; let $\phi : A \to A$ be the shift endomorphism: $\phi(x) = 1 \otimes x$. Then $\omega_{\phi^k a} \to \mathrm{tr}$ in the w^*-topology when $k \to \infty$. Since this topology coincides with the topology induced by the metric $d_{D_{\{\lambda_n\}}}$, we have $d_{D_{\{\lambda_n\}}}(\mathrm{tr}, \omega_{\phi^k a}) \to 0$, $k \to \infty$. Hence there is $k \in \mathbb{N}$ such that $d_{D_{\{\lambda_n\}}}(\mathrm{tr}, \omega_{\phi^k a}) < d_{D_{\{\lambda_n\}}}(\mathrm{tr}, \omega_a)$. Since $\omega_{\phi^k a} = \omega_a \circ \sigma_{1,k+1}$ and $\sigma_{1,k+1} = \sigma_{1,2} \circ \sigma_{2,3} \circ ... \circ \sigma_{k-1,k} \circ \sigma_{k,k+1} \circ \sigma_{k-1,k} \circ ... \circ \sigma_{2,3} \circ \sigma_{1,2}$, the result follows. $\square$

Remark 3.7. Let $A = \otimes^\infty M_n$ for some $n \in \mathbb{N}$ and let $\{\lambda_i\}$ be a strictly increasing sequence of positive real numbers, $D_{\{\lambda_i\}}$ the associated Dirac operator. Then $\mathrm{Iso}(A, H_{\mathrm{tr}}, D_{\{\lambda_i\}})$ satisfies:

   (i) $\sigma_{i,j} \notin \mathrm{Iso}(A, H_{\mathrm{tr}}, D_{\{\lambda_i\}})$ for every i, j;

   (ii) $\sigma_{i,j}$ is in the normalizer of $\mathrm{Iso}(A, H_{\mathrm{tr}}, D_{\{\lambda_i\}})$ in $\mathrm{Aut}(A)$ for every i, j;

   (iii) if $\alpha \in \mathrm{Aut}(A)$ then $\alpha \in \mathrm{Iso}(A, H_{\mathrm{tr}}, D_{\{\lambda_i\}})$ if and only if $\mathrm{id}_{M_n} \otimes \alpha \in \mathrm{Iso}(A, H_{\mathrm{tr}}, D_{\{\lambda_i\}})$.

We will see below how a stronger version of Lemma 3.6 can be deduced in the concrete case of the CAR-algebra. Before specializing, we prove a result for the case of more general UHF-algebras.

Let $A = \otimes^\infty M_k$ with connecting morphisms $a \mapsto a \otimes 1$ be a UHF-algebra of type k^∞ and let H_{tr} be the GNS-Hilbert space associated to the unique trace tr; H_{tr} is an inductive

limit of finite-dimensional Hilbert spaces H_n associated to M_{k^n}, $n \in \mathbb{N}$. Let (A, H_{tr}, D) be the Christensen-Ivan spectral triple associated to an increasing sequence of real numbers $\{\lambda_n\}$. If $v \in H_{\mathrm{tr}}$ is a unit vector we denote by ϕ_v the associated vector state on A. Denote $A_n = M_{k^n} \simeq M_k^{\otimes n}$.

Lemma 3.8. *Keep the above notation and let $n \in \mathbb{N}$, $v \in H_n$ with $\mathrm{tr}(v^*v) = 1$. Then*

$$d_D(\phi_v, \mathrm{tr}) = \sup_{x \in A_n,\, \|[D,x]\| \leqslant 1} |\operatorname{tr}(x) - \phi_v(x)|.$$

Proof. It is enough to show that for every $\tilde{x} \in \bigcup_h A_h$ with $\|[D, \tilde{x}]\| \leqslant 1$ there is $x \in A_n$ such that $\|[D, x]\| \leqslant 1$ and $|\operatorname{tr}(x) - \phi_v(x)| = |\operatorname{tr}(\tilde{x}) - \phi_v(\tilde{x})|$. Hence let $\tilde{x} \in \bigcup_h A_h$ and consider the conditional expectation $E_n : \bigcup_h A_h \to A_n$ given by $E_n(\sum x_1 \otimes x_2 \otimes ... \otimes x_n \otimes x_{n+1} \otimes ...) = \sum x_1 \otimes x_2 \otimes ... \otimes x_n \otimes \operatorname{tr}(x_{n+1}) \otimes \operatorname{tr}(x_{n+2}) \otimes ....$ We claim that $E_n(\tilde{x}) \in A_n$ is the desired element. For note that for every $v_1, v_2 \in H_n$ we have

$$\langle v_1, E_n(\tilde{x})v_2 \rangle = \langle v_1, \tilde{x}v_2 \rangle \tag{1}$$

and so, denoting by P_n the orthogonal projection from H_{tr} onto H_n, we have

$$P_n \tilde{x} P_n = P_n E_n(\tilde{x}) P_n.$$

Moreover, $E_n(\tilde{x})$ leaves every H_k invariant for $k \geqslant n$. Hence

$$[D, E_n(\tilde{x})] = \sum_k \alpha_k (Q_k E_n(\tilde{x}) - E_n(\tilde{x})Q_k) = \sum_{k=1}^n \alpha_k P_n [Q_k, E_n(\tilde{x})] P_n = P_n [D, \tilde{x}] P_n,$$

from which we obtain $\|[D, E_n(\tilde{x})]\| \leqslant \|[D, \tilde{x}]\|$. Furthermore, it follows again from (1) that $|\operatorname{tr}(\tilde{x}) - \phi_v(\tilde{x})| = |\operatorname{tr}(E_n(\tilde{x})) - \phi_v(E_n(\tilde{x}))|$. $\square$

The above Proposition is the key observation for the explicit computation of the distance between the trace and certain states in the case of the CAR-algebra. This computation reveals the impossibility for the automorphisms $\sigma_{i,j}$ (see the discussion before Lemma 3.6) to belong to the ISO-group of a Christensen-Ivan spectral triple. We recall that M_2 is linearly generated by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to clarify the next result, we first illustrate a concrete basic example in which the distance of the trace from a specific vector state in the case A is the CAR-algebra can be computed. In the following we fix a Christensen-Ivan spectral triple associated to a sequence of diverging pairwise different eigenvalues for such algebra. Let

$$v = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \quad \in M_2 \otimes 1$$

considered as a norm one vector in the GNS representation H_{tr}. We have $\phi_v(\sigma_1) = \phi_v(\sigma_2) = 0$ and $\phi_v(\sigma_3) = \phi_v(\sigma_4) = 1$. We want to show that the sup appearing in the definition of the distance between ϕ_v and tr can be computed on multiples of σ_3; in order

to do so we need to show that for every $\alpha_1 = \gamma, \alpha_2 = \delta, \alpha_3 = \beta, \alpha_4 = \alpha \in \mathbb{C}$ we have $\|[D, \sum_{i=1}^4 \alpha_i \sigma_i]\| \geqslant \|[D, \alpha_3 \sigma_3]\|$. Let

$$x = \begin{pmatrix} \alpha + \beta & \gamma - i\delta \\ \gamma + i\delta & \alpha - \beta \end{pmatrix} \quad \in M_2.$$

Using the C^*-identity we see that the norm of the commutator $[D, x]$ is given by $\|[D, x]\| = \lambda_1 \max\{\|x_{0,1}\|, \|x_{1,0}\|\}$, where $x_{0,1} = P_0 x (P_1 - P_0)$ and $x_{1,0} = (P_1 - P_0) x P_0$. A computation shows that for $w \in \mathbb{C}$ we have

$$x_{1,0} w = \begin{pmatrix} \beta w & (\gamma - i\delta)w \\ (\gamma + i\delta)w & -\beta w \end{pmatrix},$$

from which we obtain

$$\|x_{1,0}\|^2 = |\beta|^2 + \frac{1}{2}(|\gamma + i\delta|^2 + |\gamma - i\delta|^2) \geqslant |\beta|^2.$$

Note now that $(\beta \sigma_3)_{0,1} : \mathbb{C}\xi_{\mathrm{tr}}^\perp \subset H_1 \to \mathbb{C}$ is given by

$$(\beta\sigma_3)_{0,1} \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & -w_{1,1} \end{pmatrix} = \beta w_{1,1}$$

and so $\|(\beta\sigma_3)_{0,1}\| = |\beta|$. Similarly we have $\|(\beta\sigma_3)_{1,0}\| = |\beta|$. In particular $\|x_{1,0}\| \geqslant \max\{\|(\beta\sigma_3)_{1,0}\|, \|(\beta\sigma_3)_{0,1}\|\}$ and so $\|[D, x]\| \geqslant \|[D, \beta\sigma_3]\|$. It follows from Lemma 3.8 that

$$d_D(\phi_v, \mathrm{tr}) = \sup_{\beta \in \mathbb{C}\,:\,\|[D,\beta\sigma_3]\| \leqslant 1} |\mathrm{tr}(\beta\sigma_3) - \phi_v(\beta\sigma_3)| = \sup_{\beta \in \mathbb{C}\,:\,\|[D,\beta\sigma_3]\| \leqslant 1} |\beta|.$$

In order to obtain this value we only need the value of $\|[D, \beta\sigma_3]\|$, which we already computed and is given by $|\beta|\lambda_1$ and so

$$d_D(\phi_v, \mathrm{tr}) = \lambda_1^{-1}.$$

This specific example already contains all the ideas needed for the general case.

Proposition 3.9. *Let $A = \otimes^\infty M_2$, for $l \in \{1, 2, 3\}$ denote $B_l := \{v \in M_2 \mid \phi_v(\sigma_l) = \phi_v(\sigma_4) = 1, \ \phi_v(\sigma_j) = 0 \text{ for } j \in \{1, 2, 3\}\setminus\{l\}\} \subset M_2$ (ϕ_v is the vector state associated to v: $\phi_v(\cdot) = \mathrm{tr}(v^*(\cdot)v)$) and let $v \in \mathrm{ad}(U(M_2))(B_l)$ for some l. For every $n \in \mathbb{N}$ we have*

$$d(\phi_{1^{\otimes n} \otimes v}, \mathrm{tr}) = \frac{1}{\lambda_{n+1}}.$$

The supremum for the distance is attained on the element $1^{\otimes n} \otimes \sigma_l / \|[D, 1^{\otimes n} \otimes \sigma_l]\| \in 1^{\otimes n} \otimes M_2 \subset A_{n+1}$.

Proof. First of all note that we can suppose $v \in B_l$, since $\mathrm{ad}(1^{\otimes n} \otimes U(M_2)) \subset \mathrm{Iso}(A, D, H_{\mathrm{tr}})$ in virtue of Theorem 3.1. Let σ_i, $i = 1, ..., 4$ be the Pauli matrices and v be as in the statement. For every

$$x = \sum_{i_1, i_2, ..., i_{n+1}=1}^4 \alpha_{i_1, i_2, ..., i_{n+1}} \sigma_{i_1} \otimes \sigma_{i_2} \otimes ... \otimes \sigma_{i_{n+1}} \in A_{n+1}$$

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we denote

$$\tilde{x} = x - \sum_{i_1,i_2,\ldots,i_n=1}^{4} \alpha_{i_1,i_2,\ldots,i_n,4}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n} \otimes 1 \qquad \in A_{n+1}.$$

We want to show that

$$d(\phi_{1^{\otimes n}\otimes v}, \mathrm{tr}) = \sup_{x\in A_{n+1}, \|[D,x]\|\leqslant 1} |\phi_{1^{\otimes n}\otimes v}(x) - \mathrm{tr}(x)| = \sup_{x\in A_{n+1}, \|[D,\tilde{x}]\|\leqslant 1} |\phi_{1^{\otimes n}\otimes v}(\tilde{x})|$$

$$= \sup_{x\in A_{n+1}, \|[D,\tilde{x}]\|\leqslant 1} |\alpha_{4,4,\ldots,4,l}|.$$

For note that $\phi_{1^{\otimes n}\otimes v}(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}) = \prod_{k=1}^{n} \delta_{i_k,4}(\delta_{i_{n+1},l} + \delta_{i_{n+1},4})$ and $\mathrm{tr}(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}) = \prod_{k=1}^{n+1} \delta_{i_k,4}$. Hence for every $x = \sum_{i_1,i_2,\ldots,i_{n+1}=1}^{4} \alpha_{i_1,i_2,\ldots,i_{n+1}}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}$ we have $|\phi_{1^{\otimes n}\otimes v}(x) - \mathrm{tr}(x)| = |\alpha_{4,4,\ldots,4,l}|$. We are then left to show that for every $x \in A_{n+1}$, $\|[D,x]\| \geqslant \|[D,\tilde{x}]\|$. In order to do so, let $(i_1, i_2, \ldots, i_{n+1}) \in \{1,2,3,4\}^{\times n} \times \{1,2,3\}$ and $w \in H_n$. Then $P_n\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}w = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_n}w \otimes \mathrm{tr}(\sigma_{i_{n+1}}) = 0$ and so $P_n\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}P_n = 0$, which gives $P_n[D,\sigma_{i_1} \otimes \sigma_{i_2} \otimes \ldots \otimes \sigma_{i_{n+1}}]P_n = 0$ so that $P_n[D,\tilde{x}]P_n = 0$. On the other hand, $x - \tilde{x} \in A_n$ commutes with Q_k for every $k > n$; it follows that $[D, x - \tilde{x}]Q_k = Q_k[D, x - \tilde{x}] = 0$ for each such k. Also, a direct computation shows

$$Q_{n+1}[D,\tilde{x}]Q_{n+1} = Q_{n+1}(\sum_i \alpha_i Q_i\tilde{x} - \sum_i \alpha_i\tilde{x}Q_i)Q_{n+1}$$

$$= \alpha_{n+1}Q_{n+1}\tilde{x}Q_{n+1} - \alpha_{n+1}Q_{n+1}\tilde{x}Q_{n+1} = 0.$$

Thus $[D,\tilde{x}]$ and $[D, x - \tilde{x}]$ satisfy the following properties:

$$P_n[D,\tilde{x}]P_n = 0, \qquad P_{n+1}[D,\tilde{x}]P_{n+1} = [D,\tilde{x}], \qquad Q_{n+1}[D,\tilde{x}]Q_{n+1} = 0,$$

$$[D, x - \tilde{x}]Q_k = Q_k[D, x - \tilde{x}] = 0 \qquad \forall k > n.$$

Hence we want to check that for every $y, z \in \mathbb{B}(H_{\mathrm{tr}})$ with $P_{n+1}yP_{n+1} = y$, $P_nyP_n = Q_{n+1}yQ_{n+1} = 0$ and $zQ_k = Q_kz = 0$ for every $k > n$ we have $\|y\| \leqslant \|y + z\|$. Under these assumptions we have $\|y|_{K_i}\| \leqslant \|(y + z)|_{K_i}\|$ for every $i = 0, 1, \ldots, n+1$. Now, $y = P_nyQ_{n+1} + Q_{n+1}yP_n$ and so

$$y^*y = (Q_{n+1}y^*P_n + P_ny^*Q_{n+1})(P_nyQ_{n+1} + Q_{n+1}yP_n)$$

$$= Q_{n+1}y^*P_nyQ_{n+1} + P_ny^*Q_{n+1}yP_n$$

so that $\|y\| = \max\{\|P_nyQ_{n+1}\|, \|Q_{n+1}yP_n\|\}$. We have $P_nyQ_{n+1} = y|_{K_{n+1}}$ and so $\|P_nyQ_{n+1}\| \leqslant \|y + z\|$. Now note that the above conditions for y and z are preserved under taking adjoints and so $\|y^*|_{K_{n+1}}\| \leqslant \|(y + z)^*|_{K_{n+1}}\| \leqslant \|(y + z)^*\| = \|y + z\|$. But again $y^*|_{K_{n+1}} = P_ny^*Q_{n+1} = (Q_{n+1}yP_n)^*$, which entails $\|y^*|_{K_{n+1}}\| = \|Q_{n+1}yP_n\|$.

The next step is to show that for every $x \in A_{n+1}$ as above we have $|\alpha_{4,4,\ldots,4,l}| \leqslant \|P_0\tilde{x}Q_{n+1}\| \leqslant \lambda_{n+1}^{-1}$. The first inequality is obtained by evaluating $P_0\tilde{x}Q_{n+1}$ on the unit vector $1^{\otimes n}\otimes\sigma_l \in K_{n+1}$, which gives, using the orthogonality relations for the Pauli matrices, $P_0\tilde{x}(1^{\otimes n}\otimes\sigma_l) = \alpha_{4,4,\ldots,4,l}$. For what concerns the second inequality, note that

$$[D,\tilde{x}] = \lambda_{n+1}Q_{n+1}\tilde{x}P_0 + (\lambda_{n+1} - \lambda_1)Q_{n+1}\tilde{x}Q_1 + \ldots + (\lambda_{n+1} - \lambda_n)Q_{n+1}\tilde{x}Q_n -$$

$$\lambda_{n+1}P_0\tilde{x}Q_{n+1} - (\lambda_{n+1} - \lambda_1)Q_1\tilde{x}Q_{n+1} - \ldots - (\lambda_{n+1} - \lambda_n)Q_n\tilde{x}Q_{n+1}.$$

Since $\|[D, \tilde{x}]\| \leq 1$ we get $\|P_0 \tilde{x} Q_{n+1}\| \leq \lambda_{n+1}^{-1}$.
We are left to prove that

$$|\phi_{1^{\otimes n} \otimes v}(1^{\otimes n} \otimes \sigma_l/\|[D, 1^{\otimes n} \otimes \sigma_l]\|) - \operatorname{tr}(1^{\otimes n} \otimes \sigma_l/\|[D, 1^{\otimes n} \otimes \sigma_l]\|)| = \lambda_{n+1}^{-1}.$$

This follows from the fact that $\|[D, 1^{\otimes n} \otimes \sigma_l]\| = \lambda_{n+1}$. For note that

$$\begin{aligned}
\|[D, 1^{\otimes n} \otimes \sigma_l]\| &= \max\{\|\lambda_{n+1} P_0 (1^{\otimes n} \otimes \sigma_l) Q_{n+1} + (\lambda_{n+1} - \lambda_n) Q_1 (1^{\otimes n} \otimes \sigma_l) Q_{n+1} + \ldots \\
&\quad + (\lambda_{n+1} - \lambda_n) Q_n (1^{\otimes n} \otimes \sigma_l) Q_{n+1}\|, \|\lambda_{n+1} Q_{n+1}(1^{\otimes n} \otimes \sigma_l) P_0 \\
&\quad + (\lambda_{n+1} - \lambda_1) Q_{n+1} (1^{\otimes n} \otimes \sigma_l) Q_1 + \ldots + (\lambda_{n+1} - \lambda_n) Q_{n+1} (1^{\otimes n} \otimes \sigma_l) Q_n\|\} \\
&= \|\lambda_{n+1} P_0 (1^{\otimes n} \otimes \sigma_l) Q_{n+1} + (\lambda_{n+1} - \lambda_n) Q_1 (1^{\otimes n} \otimes \sigma_l) Q_{n+1} + \ldots \\
&\quad + (\lambda_{n+1} - \lambda_n) Q_n (1^{\otimes n} \otimes \sigma_l) Q_{n+1}\| \\
&= \|\lambda_{n+1} Q_{n+1} (1^{\otimes n} \otimes \sigma_l) P_0 + (\lambda_{n+1} - \lambda_1) Q_{n+1} (1^{\otimes n} \otimes \sigma_l) Q_1 + \ldots \\
&\quad + (\lambda_{n+1} - \lambda_n) Q_{n+1} (1^{\otimes n} \otimes \sigma_l) Q_n\|.
\end{aligned}$$

Let $\alpha \oplus v_1 \oplus v_2 \oplus \ldots \oplus v_n \in H_0 \oplus K_1 \oplus \ldots \oplus K_n$ with $|\alpha|^2 + \sum_{i=1}^n \|v_i\|^2 = 1$. We have

$$\begin{aligned}
\|(\lambda_{n+1} Q_{n+1}(1^{\otimes n} &\otimes \sigma_l) P_0 + (\lambda_{n+1} - \lambda_1) Q_{n+1}(1^{\otimes n} \otimes \sigma_l) Q_1 + \ldots \\
&+ (\lambda_{n+1} - \lambda_n) Q_{n+1}(1^{\otimes n} \otimes \sigma_l) Q_n)(\alpha \oplus v_1 \oplus \ldots \oplus v_n)\|^2 \\
&= \lambda_{n+1}^2 |\alpha|^2 + (\lambda_{n+1} - \lambda_1)^2 \|v_1\|^2 + \ldots + (\lambda_{n+1} - \lambda_n)^2 \|v_n\|^2 \\
&\leq \lambda_{n+1}^2 (|\alpha|^2 + \sum_{i=1}^n \|v_i\|^2) = \lambda_{n+1}^2,
\end{aligned}$$

which gives $\|\lambda_{n+1} Q_{n+1}(1^{\otimes n} \otimes \sigma_l) P_0 + (\lambda_{n+1} - \lambda_1) Q_{n+1}(1^{\otimes n} \otimes \sigma_l) Q_1 + \ldots + (\lambda_{n+1} - \lambda_n) Q_{n+1}(1^{\otimes n} \otimes \sigma_l) Q_n\| = \lambda_{n+1}$. $\square$

Corollary 3.10. *Let $A = \otimes^\infty M_2$ and suppose that the eigenvalues λ_n of the Dirac operator are pairwise distinct. Let $\alpha \in \operatorname{Aut}(A)$ be given with the following property: there exist $k \neq m \in \mathbb{N}$, $l, i \in \{1, 2, 3\}$, $v \in \operatorname{ad}(U(M_2))(B_l)$ such that $\alpha(1^{\otimes m} \otimes v) \in 1^{\otimes k} \otimes \operatorname{ad}(U(M_2))(B_i)$. Then α does not belong to $\operatorname{ISO}(A, H_{\mathrm{tr}}, D)$. In particular this applies to the automorphisms $\sigma_{i,j}$ given by permutation of the i-th and j-th tensor factors.*

In the end of this section we include a related result the authors think is of independent interest.

Proposition 3.11. *Denote by $\varphi : \bigcup_n A_n \to \bigcup_n A_n$ the endomorphism given by $\varphi(y) = 1 \otimes y$. For every $n \in \mathbb{N}$, $c(n) > 0$ such that $(c(n) + 1)\lambda_n \leq \lambda_{n+1}$ we have*

$$\|[D, x]\| \leq c(n)^{-1} \lambda_n^{-1} \lambda_1 \|[D, \varphi^n(x)]\|$$

for every $x \in A_1$. In particular if $2\lambda_1 < \lambda_2$, then $\|[D, \varphi(x)]\| \geq \frac{\lambda_2 - \lambda_1}{\lambda_1}\|[D, x]\|$ for very $x \in A_1$, with $\frac{\lambda_2 - \lambda_1}{\lambda_1} > 1$.

Proof. Let $x \in A_1$ and ξ be the cyclic vector. First we show that $\|[D, x]\| \leq \lambda_1 \|Q_1 x \xi\|_2$. For note that the C^*-identity gives $\|[D, x]\| = \lambda_1 \max\{\|P_0 x Q_1\|, \|Q_1 x P_0\|\}$. Now, for every $v \in Q_1 H$ we have $P_0 x v = \operatorname{tr}(xa)\xi = \operatorname{tr}((x - \operatorname{tr}(x))a)\xi$, where $a\xi = v$; hence $\|P_0 x Q_1\| \leq$

$\|Q_1 x\xi\|_2$. Similarly, for $\alpha \in \mathbb{C}$, $Q_1 x\alpha\xi = \alpha(x - \operatorname{tr}(x))\xi$ and so again $\|Q_1 x P_0\| \leqslant \|Q_1 x\xi\|_2$. A direct computation gives $P_n[D, \varphi^n(x)]P_n = 0$; hence

$$P_n[D, \varphi^n(x)]^*[D, \varphi^n(x)]P_n = P_n[D, \varphi^n(x)]^* Q_{n+1}[D, \varphi^n(x)]P_n$$

and we only need to show that $\|P_n[D, \varphi^n(x)]^* Q_{n+1}[D, \varphi^n(x)]P_n\| \geqslant c(n)^2 \lambda_n^2 \|Q_1 x\xi\|_2^2$. The operator $P_n[D, \varphi^n(x)]^* Q_{n+1}[D, \varphi^n(x)]P_n$ is explicitly given by

$$P_n[D, \varphi^n(x)]^* Q_{n+1}[D, \varphi^n(x)]P_n = \sum_{i,j=0}^{n} (\lambda_{n+1} - \lambda_i)(\lambda_{n+1} - \lambda_j) Q_i(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) Q_j$$

$$= \sum_{i=0}^{n}(\lambda_{n+1} - \lambda_i)^2 Q_i(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) Q_i$$

$$+ \sum_{i<j}(\lambda_{n+1} - \lambda_i)(\lambda_{n+1} - \lambda_j)[Q_i(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) Q_j + Q_j(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) Q_i]$$

$$\geqslant c(n)^2 \lambda_n^2 (P_n(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) P_n).$$

Explicit computations give, for $v = a\xi \in H_n$, $P_n(1^{\otimes n} \otimes x^*) Q_{n+1}(1^{\otimes n} \otimes x) P_n(v) = a \otimes \operatorname{tr}(x^*(x - \operatorname{tr}(x)))\xi = a \otimes \|Q_1 x\xi\|_2^2$, from which we obtain $\|P_n(1^{\otimes n} \otimes x)^* Q_{n+1}(1^{\otimes n} \otimes x) P_n\| \geqslant c(n)^2 \lambda_n^2 \|Q_1 x\xi\|_2^2$ and

$$\|[D, \varphi^n(x)]\|^2 \geqslant \|P_n[D, \varphi^n(x)]^*[D, \varphi^n(x)]P_n\| \geqslant c(n)^2 \lambda_n^2 \|Q_1 x\xi\|_2^2 \geqslant c(n)^2 \lambda_n^2 \lambda_1^{-2} \|[D, x]\|^2. \qquad \square$$

3.1 The case of the Cantor set

Let X be a (compact) Cantor set. There is a nested partition $\mathcal{P} = \{\mathcal{P}_i\}$ with $\mathcal{P}_i = \{U_{i,j}, \ j \in \mathbb{Z}_2^i\}$ of X consisting of clopen sets such that for every $i \in \mathbb{N}$, $j \in \mathbb{Z}_2^i$ we have $U_{i,j} = U_{i+1,j\oplus 0} \sqcup U_{i+1,j\oplus 1}$ and every continuous function on X is the uniform limit of $\mathcal{P}$-simple functions. If we identify X with $\prod_{j=1}^{\infty}\{0, 1\}$ (with the product topology), an explicit description of such a nested partition can be given in the following way: for every $l \in \mathbb{N}$ let π_l be the projection on the factor $X_l = \prod_{j=0}^{l}\{0, 1\}$ of $\prod_{j=0}^{\infty}\{0, 1\}$; given a finite word $w \in X_l$ denote by U_w the clopen set given by $U_w := \{x \in X \mid \pi_l(x) = w\}$ and define $\mathcal{P}_i = \{U_w \mid w \in X_i\}$. The C^*-algebra of continuous functions on X is the inductive limit of the sequence of finite dimensional C^*-algebras $\mathbb{C}^{2^i}$ consisting of $\mathcal{P}_i$-simple functions, where the connecting morphism $\mathbb{C}^{2^i} \to \mathbb{C}^{2^{i+1}}$ is given by considering a $\mathcal{P}_i$-simple function as a $\mathcal{P}_{i+1}$-simple function. This gives a concrete realization of $C(X)$ as the AF-algebra $\lim_i \mathbb{C}^{2^i}$ with connecting morphisms $\phi_{i,i+1}(\oplus_{k=1}^{2^i} a_k) = \oplus_{k=1}^{2^i}(a_k \oplus a_k)$. Fix such a nested partition $\mathcal{P}$.

Proposition 3.12. *Let X be a (compact) Cantor set and let μ be the uniform measure on X, viewed as a faithful state on $C(X)$. Let $(C(X), H_\mu, D)$ be the Christenen-Ivan spectral triple associated with a non-degenerate sequence $\{\lambda_n\}$ and to the AF-structure induced by $\mathcal{P}$. Then, denoting by $S_2 \simeq \mathbb{Z}_2$ the group of permutations of the set $\{0, 1\}$, we have*

$$\operatorname{Iso}(C(X), H_\mu, D) = \prod_{i=1}^{\infty} S_2^{2^{i-1}}$$

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Proof. Let $\alpha \in \mathrm{Iso}(C(X), H_\mu, D)$. Since α respects the filtration of the algebra (by Theorem 3.1), for every $i \in \mathbb{N}$ it induces an automorphism on $\mathbb{C}^{2^i}$ which leaves $\oplus^{2^{i-1}}(\mathrm{diag}(\mathbb{C}))$ invariant; by Gelfand duality $\alpha|_{\mathbb{C}^{2^i}}$ is an element of $S_2^{2^{i-1}} \times G_i$, where G_i is the group of automorphisms of $\oplus^{2^{i-1}}(\mathrm{diag}(\mathbb{C}))$ which respects the filtration up to the $(i-1)$-th step. Hence $\alpha|_{\mathbb{C}^{2^i}}$ belongs to $\prod_{k=1}^{i} S_2^{2^{k-1}}$. Clearly every element of $\prod_{i=1}^{\infty} S_2^{2^{i-1}}$ induces an automorphism α of $C(X)$ which belongs to $\mathrm{Iso}(C(X), H_\mu, D)$. $\qquad\square$

Proposition 3.13. *Let X be a (compact) Cantor set and consider the same setting as in Proposition 3.12, in the particular case of a Dirac operator associated to a sequence of real numbers $\{\lambda_n\}$ with $\lambda_n = \gamma^{-n+1}$, where $0 < \gamma < (3 - \sqrt{5})/2$. Then*

$$\mathrm{ISO}(C(X), H_\mu, D) = \mathrm{Iso}(C(X), H_\mu, D).$$

Proof. Let α be an automorphism of $C(X)$ (by an abuse of notation we will eventually consider α as a homeomorphism of X) which leaves the metric d_γ induced by D invariant. Following [4], for $x, y \in X$ let $m(x, y)$ be the least integer n satisfying $x(n) \neq y(n)$, where we view x and y as elements of $\prod \mathbb{Z}_2$. By Theorem 4.1 (ii) of [4], under the choice of γ as in the statement, if $x, y \in X$ satisfy $m(\alpha x, \alpha y) \neq m(x, y)$, then $d_\gamma(\alpha x, \alpha y) \neq d_\gamma(x, y)$; hence, if α leaves d_γ invariant, then it also leaves m invariant. We will see that this condition guarantees $\alpha(A_n) \subset A_n$ for every n. First of all note that, since the image under α of a clopen set is a clopen set, for every $n \in \mathbb{N}$ the image of a minimal projection $p \in A_n$ under α is a finite sum of minimal projections in $\bigcup_i A_i$. Suppose now that there are integers $n < k$ such that the image of $\alpha|_{A_n}$ contains an element b of A_k which is not in A_n. If we write $b = \oplus_{i=1}^{2^n}(\oplus_{j=1}^{2^{k-n}} a_{i,j})$, the condition that b does not belong to A_n is equivalent to the existence of $i \in \{1, ..., 2^n\}$ and $j_1, j_2 \in \{1, ..., 2^{k-n}\}$ such that $a_{i,j_1} \neq a_{i,j_2}$; it follows that there are two characters x and y with $m(x, y) > n$ and such that $x|_{A_n} = y|_{A_n}$ and $x \circ \alpha|_{A_n} \neq y \circ \alpha|_{A_n}$, which gives $m(\alpha x, \alpha y) \leqslant n$. $\qquad\square$

4 Isometries of crossed products

In this section we show that, under suitable hypothesis, given an action of a discrete group on a C^*-algebra, the crossed product automorphisms given in [7] are elements of the Iso-group of the crossed product spectral triple introduced in [10].

Let A be a unital C^*-algebra, G a countable discrete group and $\alpha : G \to \mathrm{Aut}(A)$ an action. As in [7] let $Z^1(G, U(A)) := \{c : G \to U(A) \mid c(gh) = c(g)\alpha_g(c(h)) \; \forall g, h \in G\}$. We consider the set $\mathcal{G}(A, \alpha) := \{(c, \beta, \sigma) \mid c \in Z^1(G, U(A)), \beta \in \mathrm{Aut}(A), \sigma \in \mathrm{Aut}(G), \; \beta \circ \alpha_g = \mathrm{ad}\, c_{\sigma(g)} \circ \alpha_{\sigma(g)} \circ \beta\} \subset Z^1(G, U(A)) \times \mathrm{Aut}(A) \times \mathrm{Aut}(G)$, endowed with the group structure $(c, \beta, \sigma)(c', \beta', \sigma') = (\beta(c' \circ \sigma^{-1}) \cdot c, \beta \circ \beta', \sigma \circ \sigma')$, $(c, \beta, \sigma)^{-1} = (\beta^{-1}(c^* \circ \sigma), \beta^{-1}, \sigma^{-1})$ (see [7]). There is a group homomorphism $\Phi : \mathcal{G}(A, \alpha) \to \mathrm{Aut}(A \rtimes_{\alpha, r} G, A)$ explicitly given by $\Phi(c, \beta, \sigma)(\sum_{g \in G} a_g \lambda_g) = \sum_{g \in G} \beta(a_g) c_{\sigma(g)} \lambda_{\sigma(g)}$ (the fact that this is actually an automorphism of the reduced crossed product follows since it is continuous in the L^1-norm). As shown in [7], under suitable assumptions $\Phi(\mathcal{G}(A, \alpha)) = \mathrm{Aut}(A \rtimes_r G, A)$.

If (A, H, D) is a spectral triple and $\mathcal{A}$ is a dense $*$-subalgebra of A such that $[D, a]$ extends to a bounded operator on H for every $a \in \mathcal{A}$, we will say that $(\mathcal{A}, H, D)$ is a spectral triple for A. This notion coincides with the notion of odd spectral triple

considered in [10] Definition 2.1. In [10] Theorem 2.7 it is proved that if $(\mathcal{A}, H, D)$ is a spectral triple for A and G is a discrete countable group acting on A equipped with a proper translation bounded function $l : G \to \mathbb{Z}$ ([10, Example 2.4]) then, under some additional hypotheses (see the statement of Theorem 4.1 below), $(C_c(G, \mathcal{A}), (H \otimes l^2(G))^{\oplus 2}, D_l)$ is a spectral triple for $A \rtimes_r G$, where D_l is given by

$$D_l = \begin{pmatrix} 0 & D \otimes 1 - i \otimes M_l \\ D \otimes 1 + i \otimes M_l & 0 \end{pmatrix}$$

and M_l is the usual self-adjoint extension of the unbounded operator of multiplication by l on $l^2(G)$.

Theorem 4.1. *Let A be a unital C^*-algebra endowed with an action α of a countable discrete group G, equipped with a proper translation bounded function $l : G \to \mathbb{Z}$. Suppose we are given a spectral triple $(\mathcal{A}, H, D)$ for A which satisfies the hypothesis of Theorem 2.7 of [10], namely: $\alpha_g(\mathcal{A}) \subset \mathcal{A}$ for every $g \in G$, $\sup_{g \in G} \|[D, \alpha_g(a)]\| < \infty$ for every $a \in \mathcal{A}$. Let $(c, \beta, \sigma) \in \mathcal{G}(A, \alpha)$ be such that $\beta \in \mathrm{Iso}(A, H, D)$, $l \circ \sigma = l$ and $c_g \in \mathbb{C} \cdot 1_A$ for every $g \in G$. Then $\Phi(c, \beta, \sigma) \in \mathrm{Iso}(A \rtimes_{\alpha, r} G, (H \otimes \ell^2(G))^{\oplus 2}, D_l)$, where D_l is the above Dirac operator.*

Proof. Let $U_\beta \in U(H)$ be a unitary implementing the automorphism β. Let $U_{(c\beta,\sigma)}$ be the unitary operator in $U(H \otimes \ell^2(G))$ given, for $\xi \in H$ and $g \in G$, by $U_{(c,\beta,\sigma)}(\xi \otimes \delta_g) = (c^*_{\sigma(g^{-1})} U_\beta \xi) \otimes \delta_{\sigma(g)}$. For $a \in A$, $\xi \in H$ and $g, h \in G$ we compute

$$\Phi(c, \beta, \sigma)(a\lambda_g)U_{(c,\beta,\sigma)}\xi \otimes \delta_h = (\beta(a)c_{\sigma(g)}\lambda_{\sigma(g)})((c^*_{\sigma(h^{-1})} U_\beta \xi) \otimes \delta_{\sigma(h)})$$
$$= (\alpha_{\sigma(gh)^{-1}}(\beta(a)c_{\sigma(g)})c^*_{\sigma(h^{-1})} U_\beta \xi) \otimes \delta_{\sigma(gh)}$$
$$= (c^*_{\sigma(gh)^{-1}}\beta(\alpha_{(gh)^{-1}}(a))c_{\sigma(gh)^{-1}}\alpha_{\sigma(gh)^{-1}}(c_{\sigma(g)})c^*_{\sigma(h)^{-1}} U_\beta \xi) \otimes \delta_{\sigma(gh)}$$
$$= (c^*_{\sigma(gh)^{-1}}\beta(\alpha_{(gh)^{-1}}(a))c_{\sigma(h)^{-1}}c^*_{\sigma(h)^{-1}} U_\beta \xi) \otimes \delta_{\sigma(gh)}$$
$$= (c^*_{\sigma(gh)^{-1}}\beta(\alpha_{(gh)^{-1}}(a))U_\beta \xi) \otimes \delta_{\sigma(gh)}.$$

On the other hand

$$U_{(c,\beta,\sigma)}a\lambda_g(\xi \otimes \delta_h) = U_{(c,\beta,\sigma)}(\alpha_{(gh)^{-1}}(a)\xi) \otimes \delta_{gh}$$
$$= (c^*_{\sigma(gh)^{-1}} U_\beta \alpha_{(gh)^{-1}}(a)\xi) \otimes \delta_{\sigma(gh)} = (c^*_{\sigma(gh)^{-1}}\beta(\alpha_{(gh)^{-1}}(a))U_\beta \xi) \otimes \delta_{\sigma(gh)}.$$

Hence $U_{(c,\beta,\sigma)}$ implements the automorphism $\Phi(c, \beta, \sigma)$ in $H \otimes \ell^2(G)$. Note that in the above computation we did not use the fact that c is a character. Represent now $A \rtimes_{\alpha, r} G$ diagonally on $(H \otimes \ell^2(G))^{\oplus 2}$); then $\Phi(c, \beta, \sigma)$ is unitarily implemented by $U_{(c,\beta,\sigma)} \otimes 1_2$ in this representation. Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the eigenvalues of D and $\{P_k\}$ the corresponding spectral projections. Under the identification $H \otimes \ell^2(G) \simeq \ell^2(G, H)$, we can write $\xi \in (H \otimes \ell^2(G))^{\oplus 2}$ as $(\xi_g)_{g \in G}$, with $\xi_g = (\xi_1, \xi_2)_g \in H \oplus H$ for every $g \in G$ and the domain of the unbounded operator D_l is given by the set of vectors $\xi = (\xi_1, \xi_2) \in (H \otimes l^2(G))^{\oplus 2}$ which satisfy

$$\sum_{g \in G} \sum_{k \in \mathbb{N}} (\lambda_k^2 + l(g)^2)\frac{1}{4} \left\| \begin{pmatrix} P_k\xi_1 + \frac{\lambda_k - il(g)}{\sqrt{\lambda_k^2 + l(g)^2}}P_k\xi_2 \\ \frac{\lambda_k + il(g)}{\sqrt{\lambda_k^2 + l(g)^2}}P_k\xi_1 + P_k\xi_2 \end{pmatrix}_g \right\|^2 < \infty.$$

Under the hypothesis that U_β commutes with the spectral projections of D, that c is a character and that $l \circ \sigma = l$, this relation is left invariant under $U_{(c,\beta,\sigma)} \otimes 1_2$. In order

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to have that $[D_l, U_{(c,\beta,\sigma)}] = 0$ it is enough to check that $[D, c_g^* U_\beta] = 0$ for every $g \in G$ and that $[1 \otimes M_l, U_{(c,\beta,\sigma)}] = 0$; since c is a character, these follow from $[D, U_\beta] = 0$ and $l \circ \sigma = l$ respectively. $\qquad\square$

Remark 4.2. Let A be a unital C^*-algebra endowed with an action of a countable discrete group G, $l : G \to \mathbb{Z}$ a proper translation bounded function and denote $\mathcal{G}_{(l,D,H)}(A, \alpha) :=$ $\{(c, \beta, \sigma) \in \mathcal{G}(A, \alpha) \mid l \circ \sigma = l, \ c \in \mathrm{Char}\, G, \ \beta \in \mathrm{Iso}(A, H, D)\}$. Let $(\mathcal{A}, H, D)$ be a spectral triple for A in the sense of [10] Definition 2.1 and suppose that $\alpha_g(\mathcal{A}) \subset \mathcal{A}$ for every $g \in G$, $\sup_{g \in G} \|[D, \pi_\psi(\alpha_g(a))]\| < \infty$ for every $a \in \mathcal{A}$. Theorem 4.1 states that the map $\Phi : \mathcal{G}_{(l,D,H)}(A, \alpha) \to \mathrm{Aut}(A \rtimes_{\alpha,r} G, A)$ sets up an injective group homomorphism $\Phi : \mathcal{G}_{(l,D,H)}(A, \alpha) \to \mathrm{Iso}(A \rtimes_{\alpha,r} G, (H \otimes \ell^2(G))^{\oplus 2}, D_l)$.

Examples:

(i) Let A be a unital C^*-algebra endowed with the trivial action of a countable discrete group G admitting a translation bounded function $l : G \to \mathbb{Z}$ and $(\mathcal{A}, H, D)$ be a spectral triple. In this case the crossed product is just the spatial tensor product $A \otimes C_r^*(G)$ and the Dirac operator D_l is the (even) tensor product triple (cfr. the paragraph before Proposition 2.8 in [10]). Denoting by $\mathrm{Aut}_l(G)$ the subgroup of $\mathrm{Aut}(G)$ given by the automorphisms of G which preserve l, the map Φ gives a bijection between $\mathrm{Char}\, G \times \mathrm{Iso}(A, H, D) \times \mathrm{Aut}_l(G)$ and $\mathrm{Iso}(A, H, D) \times \mathrm{Iso}(C_r^*(G), M_l, \ell^2(G))$ (see [13]). This suggests that in general the product of Iso-groups should embed inside the Iso of some tensor product of spectral triples.

(ii) Let A be a separable unital C^*-algebra and $(\mathcal{A}, H, D)$ a spectral triple for A in the sense of [10] Definition 2.1. Let also $\alpha \in \mathrm{Iso}(A, H, D)$ be such that $\alpha(\mathcal{A}) \subset \mathcal{A}$ (this is always the case if A is AF and the spectral triple is the one considered by Christensen and Ivan) and $\chi \in S^1$ be a character of $\mathbb{Z}$. Then $\Phi_{(\chi,\alpha,\mathrm{id})} \in \mathrm{Iso}(A \rtimes_{\alpha,r} \mathbb{Z}, (H \otimes \ell^2(Z))^{\oplus 2}, D_l)$, where $l : \mathbb{Z} \to \mathbb{Z}$, $l(n) = n$. An example of this situation is given by the odometer actions on the (compact) Cantor set, when we consider the spectral triple of Christensen and Ivan associated to a faithful state. In this case the associated crossed products are the Bunce-Deddens algebras (c.f. [10], section 3.1).

(iii) More generally, let $(\mathcal{A}, H, D)$ be a spectral triple for a unital C^*-algebra A in the sense of [10] Definition 2.1 and consider the subgroup $\mathrm{Iso}(\mathcal{A}, H, D)$ of $\mathrm{Iso}(A, H, D)$ which consists of elements α such that $\alpha(\mathcal{A}) \subset \mathcal{A}$. Then for every countable discrete subgroup G of $\mathrm{Iso}(\mathcal{A}, H, D)$ admitting a translation bounded function l, every α in the normalizer subgroup of G in $\mathrm{Iso}(\mathcal{A}, H, D)$ such that $l = l \circ \mathrm{ad}\,\alpha$ and character χ of G we have $\Phi(\chi, \alpha, \mathrm{ad}\,\alpha) \in \mathrm{Iso}(A \rtimes_r G, (H \otimes \ell^2(G))^{\oplus 2}, D_l)$. If $\alpha \in G$, then $\Phi(\chi, \alpha, \mathrm{ad}\,\alpha) = \chi(\cdot)\,\mathrm{ad}\,\lambda_\alpha$, but in general λ_α will differ from the unitary $U_{(1,\alpha,\mathrm{ad}\,\alpha)}$ commuting with the Dirac operator D_l. This procedure can be iterated in different ways.

(iv) Let $d \in \mathbb{N}$ and $G = \mathbb{Z}^d$ or $\mathbb{F}_d$. Consider the length function on G given by a choice of generators, this is a translation bounded function. Let σ be an automorphism of G induced by a permutation of the d generators. Let now $(\mathcal{A}, H, D)$ and α be as in example (ii). Considering the action $\tilde{\alpha}$ of G on A induced from the "forgetful" homomorphism $G \to \mathbb{Z}$ which sends every generator to the single generator of $\mathbb{Z}$,

we have that for every character $\chi \in \operatorname{Char} G$, $\Phi_{(\chi,\tilde{\alpha},\sigma)}$ belongs to $\operatorname{Iso}(A \rtimes_{\tilde{\alpha},r} G, (H \otimes \ell^2(G))^{\oplus 2}, D_l)$.

5 Postface

As pointed out by a referee, it is not unnatural to equip quantum spaces with some sort of quantum symmetries. Indeed, in the literature there is a notion of 'quantum isometry group' for spectral triples (as well as for metric spaces) in the setting of (compact) quantum groups, see e.g. the monography [2]. Loosely speaking, these quantum isometry groups should be thought of as some sort of generalization of the $\operatorname{Iso}(A, H, D)$ considered in this paper, with groups replaced by quantum groups. Even more interestingly, such quantum groups for the Christensen-Ivan spectral triples have been looked at in [3] (cf. [2] Remark 5.2.2). Therein, the authors recognize the existence of some inductive limit structure on such quantum groups and present further considerations in the commutative case, notably for the algebra of continuous functions on the middle-third Cantor set. In general one would expect that the Iso-groups associated to Christensen-Ivan spectral triples for AF-algebras considered in this work can be recovered as maximal classical subgroups (see e.g. [1, 12]) of the corresponding quantum isometry groups.

We stress out the connection between Proposition 3.12 and the the computation of the quantum isometry groups appearing in [3]. In [3] it is proved that the quantum isometry group QISO^+ of a Christensen-Ivan spectral triple associated to the uniform measure on the Cantor set is an "inductive limit" of certain C^*-algebras S_n which are defined inductively by $\mathsf{S}_1 = C(\mathbb{Z}_2)$, $\mathsf{S}_{n+1} = (\mathsf{S}_n * \mathsf{S}_n) \oplus (\mathsf{S}_n * \mathsf{S}_n)$. Now, for every $n \in \mathbb{N}$, we let $\operatorname{Iso}_n(C(X), H_\mu, D)$ denote the (finite) group of automorphisms of $\mathcal{P}_n$-simple functions which respect the filtration of the algebra in the chosen direct limit decomposition. Then, as observed in the proof of Proposition 3.12, $\operatorname{Iso}_n(C(X), H_\mu, D) \simeq \prod_{k=1}^{n} S_2^{2^{k-1}} \simeq \prod_{i=1}^{2^n-1} S_2$ and $\operatorname{Iso}(C(X), H_\mu, D)$ is the projective limit of such groups. Using the fact that the abelianization of the free product of two groups is the direct product of the abelianizations, one can check that for every $n \in \mathbb{N}$ the maximal classical subgroup of S_n is $\operatorname{Iso}_n(C(X), H_\mu, D)$. The projective structure of $\operatorname{Iso}(C(X), H_\mu, D)$ is obtained by dualizing the inductive structure of QISO^+. Hence Iso is the maximal classical subgroup of QISO^+ in the sense of [12] (cf. [8] Section 5).

6 Acknowledgments

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