Mathematical Analysis 1, academic year 2019-2020, 1st term

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S Y L L A B U S

23.09.19 (Presentation of the course). Naive set theory. Operations between sets: union, intersection and complement. De Morgan's laws. Cartesian product. Relations between sets. Reflexivity, symmetry, and transitivity. Equivalence relations. Equivalence classes. Functions between (abstract) sets. Examples and counterexamples.

25.09.19 More on equivalence classes: the quotient of a set modulo an equivalence relation. Injective functions and surjective functions between abstract sets. Bijective functions. Cardinality of a set. Finite and infinite sets. Axiomatic description of the set of natural numbers (Peano axioms) $\mathbb{N}$. The principle of induction. The sum of the first n natural numbers.

26.09.19 Oder between natural numbers. Well-ordering principle. The set of integer numbers $\mathbb{Z}$. The rational field $\mathbb{Q}$ (formal definition of a rational number as an equivalence class of fractions) and its properties. Non-existence of the square root of 2 (and, more in general, of any prime number p) in $\mathbb{Q}$.

27.09.19 Axiomatic description of an ordered field $(\mathbb{F}, \leq)$. Properties. Any ordered field contains the rational field (without proof). Archimedean property. The rational field is Archimedean. $\mathbb{Q}$ is dense in any Archimedean field (with proof).

30.09.19 Open (or closed) intervals, open (or closed) half-lines; sets bounded from above and from below. Upper bounds for a bounded (from above) set. The supremum (aka least upper bound) as the smallest upper bound. The supremum is unique provided that it exists. Axiomatic description of the real field $\mathbb{R}$ as the unique ordered field in which every bounded (form above) non-empty set has supremum. First consequences: the real field is Archimedean; existence of the square root of 2 as a real number. The absolute value of a real number and its properties. Triangle inequality

02.10.19 More on the triangle inequality. Bernoulli's inequality (proof by induction): for all $n \in \mathbb{N}$ one has $(1+x)^n \geq 1+nx$ if $x \geq -1$. Subsets bounded from below; lower bounds (or minorants) for such sets. The infimum (aka greatest lower bound) of a bounded (rom below) set. Maximum and minimum of a subset. Nested intervals principle (without proof) and its consequences: binary and decimal representation of real number between 0 and 1. Countable (infinite) sets. $\mathbb{Z}$ and $\mathbb{Q}$ are both countable (sketch of the proof).

03.10.19 $\mathbb{R}$ is uncountable (Cantor's diagonalization method). The power set, 2^X, of a set X. If X is finite, the $|2^X| = 2^{|X|}$. If X is infinite then there exists no surjection from X onto 2^X (Cantor's theorem). The equality between cardinalities $|\mathbb{R}| = |2^{\mathbb{N}}|$. A couple of exercises on the induction principle: $\sum_{k=0}^{n-1}(2k-1) = n^2$ and $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$.

04.10.19 Sequences of real numbers. Converging sequences. The limit is unique. Diverging sequences. Examples.

07.10.19 Rational operations with limits: the limit of the sum, of the product and of the quotient of two converging sequences. The limit preserves the order in $\mathbb{R}$, but it may turn strict inequalities into large inequalities. Comparison theorems. The squeeze lemma. An application of the squeeze lemma: for any $a > 0$, we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

09.10.19 Monotone sequences. Monotone sequences are regular: they either diverge (if unbounded) or converge (if bounded). The sequence $\{(1 + \frac{1}{n})^n, n = 1, 2, \dots\}$ is bounded and monotone non-decreasing. Definition of Napier's constant e as $\lim_{n\to\infty}(1 + \frac{1}{n})^n$: e is known to be a *trascendental* number with $2 < e < 3$ (without proof). Algebraic numbers as roots of polynomial with integer coefficients. Examples. The extended real line $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$. First examples of indeterminate forms.

11.10.19 Accumulation points of a given set $E \subset \mathbb{R}$. The derived set $\mathcal{D}E$ or E' of a given set $E \subset \mathbb{R}$. If $F \subset \mathbb{R}$ is finite, then $\mathcal{D}F$ is empty. $\mathcal{D}\mathbb{Q} = \mathbb{R}$. Definition of an isolated point. Bolzano-Weierstrass theorem (every bounded infinite set $E \subset \mathbb{R}$ has at least one accumaltion point) and its consequences: every bounded sequence has a converging subsequence. Exercises on limits and their computatons. Definition of $n!$ for any natural $n \in \mathbb{N}$. Lemma: if $\{a_n : n \in \mathbb{N}\}$ is a sequence with $a_n \neq 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ with $|l| < 1$ then $\lim_{n\to\infty} a_n = 0$. Some applications: $\lim_{n\to\infty} \frac{n^2}{n!} = 0$, and, more in general, $\lim_{n\to\infty} \frac{n^k}{n!} = 0$ for any $k \in \mathbb{N}$.

14.10.19 Cauchy (or fundamental) sequences. Cauchy convergence criterion: a sequence is convergent if and only if it is a Cauchy sequence (with proof). Real functions of one real variable. The domain of a function assigned by means of its analytical expression as the largest subset of $\mathbb{R}$ where the expression makes sense. The integer part $[x]$ and the fractional part $\{x\}$ of a real number $x \in \mathbb{R}$. Powers and polynomials. Rational functions.

16.10.19 A review on the exponential function and its properties. Invertible functions. The inverse function f^{-1} of an invertible function f. Compositions of functions. The logarithm function as the inverse of the exponential function and its properties. The sum of the first n squares: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

17.10.19 A review on the circular functions $\sin, \cos, \tan$. Their (local) inverse functions $\arcsin, \arccos, \arctan$. Properties of the inverse circular functions: domains and monotonicity. Topological notions: interior, exterior and boundary points of a given subset E of the real line. Examples and counterexamples. The interior E°, the exterior $\mathrm{Ext}(E)$, and the boundary ∂E of a given subset $E \subset \mathbb{R}$. Some examples: $\mathbb{Q}^\circ = \mathrm{Ext}(\mathbb{Q}) = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$. Open sets and closed sets. Open intervals and open half-lines are open sets. Closed intervals and closed half-lines are closed sets. Connectedness of the real line: $\emptyset$ and $\mathbb{R}$ are the only two sets thar are both open and closed (without proof).

18.10.19 Properties of open and closed sets: unions and finite intersections of open sets are open; intersections and finite unions of closed sets are closed. Characterization of closed sets in terms of sequences: a set $E \subset \mathbb{R}$ is closed if and only if for any sequence $\{x_n : n \in \mathbb{N}\} \subset E$ with $\lim_n x_n = x$, the limit x is still in E. Corollary: E is closed if and only if it contains all of its accumulation points, i.e. $\mathcal{D}E \subset E$. Limits of functions: first definitions and examples. Vertical and horizontal asymptotes to the graph of a function.

21.10.19 Two limits: $\lim_{x\to+\infty} a^x = +\infty$. if $a > 1$ and $\lim_{x\to+\infty} a^x = 0$ if $0 < a < 1$. Right and left limit. Bridge theorem: $\lim_{x\to x_0} f(x) = l$ if and only if for any sequence $\{x_n : n \in \mathbb{N}\} \subset D_f$ such that $x_n \to x_0 \in D_f'$ one has $\lim_n f(x_n) = l$. Applications: the limit $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist, but $\lim_{x\to 0} x^\alpha \sin(\frac{1}{x}) = 0$ for any $\alpha > 0$. Uniqueness of the limit and operations with limits (without presenting the proofs as they are exactly the same as for sequences). The squeeze lemma.

23.10.19 Some remarkable limits: $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$, $\lim_{x\to 0}\frac{\tan x}{x}=1$, $\lim_{x\to\infty}(1+\frac{1}{x})^x=e$. Infinitesimals and comparison between them: the little-o notation. Reinterpretation of the above limits in terms of infinitesimals: $\sin x=x+o(x)$, $\cos x=1-\frac{x^2}{2}+o(x^2)$, $\tan x=x+o(x)$, when $x\to 0$. Odd and even functions.

24.10.19 The limit of a compound function: let f,g be two functions $f:I\subset\mathbb{R}\to\mathbb{R}$ and $g:J\subset\mathbb{R}\to\mathbb{R}$ with $f(I)\subset J$ so that $g\circ f$ is well defined; if $x_0\in I'$ and $y_o\in J'$ with $\lim_{x\to x_0}f(x)=y_0$ and $\lim_{y\to y_0}g(y)=l$, then $\lim_{x\to x_0}g(f(x))=l$ as long as $f(x)\neq y_0$ in an open neighbourhood of x_0. Applications: computation of limits by means of a change of variable. Remarkable limits: $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$; $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$; $\lim_{x\to 0}\frac{e^x-1}{x}=1$; $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$; $\lim_{x\to 0}\frac{\log_a(x+1)}{x}=\frac{1}{\ln a}$; $\lim_{x\to\infty}(1+\frac{\alpha}{x})^x=e^\alpha$, for any $\alpha\in\mathbb{R}$. The exponential function grows to infinity faster than any other power: $\lim_{x\to+\infty}\frac{e^x}{x^\alpha}=+\infty$, for any $\alpha\in\mathbb{R}$. The logarithm function grows to infinity more slowly than any other power: $\lim_{x\to+\infty}\frac{\ln x}{x^\alpha}=0$, for any $\alpha>0$. For any $\alpha>0$ the limit holds $\lim_{x\to 0^+}x^\alpha\ln x=0$.

25.10.19 Continuity at a point x_0. Continuity on an interval I. First examples of continuous functions: powers, polynomials, and rational functions. The set of all rational functions $\mathbb{R}(x)$ as an example of an ordered field that fails to be Archimedean. Continuity of all n^{th} roots (without proof for the time being, as it can be deduced from a general result we'll see later on: any invertible continuous function has a continuous inverse). Continuity of the circular functions $\sin,\cos,\tan$ in their domains. Exercises on limits.

28.10.19 Classification of discontinuities (or singularities): removable, jump, essential discontinuity. Monotone functions can only have jump discontinuities. Moreover, the discontinuities are at most countably many. Examples and exercises.

30.10.19 Existence of zeroes for continuous functions: if $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is continuous on $[a,b]$ and $f(a)f(b)<b$, then there exists at least one $c\in(a,b)$ such that $f(c)=0$. Intermediate value theorem. Exercises.

31.10.19 Maxima and minima. Weierstrass theorem: if $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is continuous on the closed bounded interval $[a,b]$, then it admits both maximum and minimum in $[a,b]$. Application: the range of a continuous function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is still an interval: $f([a,b])=[m,M]$, where $m=\min\{f(x):x\in[a,b]\}$ and $M=\max\{f(x):x\in[a,b]\}$. More exercises on limits.

04.11.19 (Sequentially) compact subsets of the real line. Characterization of compact subsets: $K\subset\mathbb{R}$ is compact if and only if it is closed and bounded. (Generalized) Weierstrass theorem: if $f:K\subset\mathbb{R}\to\mathbb{R}$ is continuous on the compact sunset K, then it admits both maximum and minimum in K. How to find the oblique asymptotes to the graph of a function, if any. Exercises.

06.11.19 Any continuous invertible function is monotone (either increasing or decreasing). A monotone function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is continuous if and only if its range, $f([a,b])$, is an interval. The inverse of a continuous invertible function is still continuous. Continuity of the inverse circular functions. Two remarkable limits: $\lim_{x\to 0}\frac{\arcsin x}{x}=1$ and $\lim_{x\to 0}\frac{\arctan x}{x}=1$. Exercises.

07.11.19 Uniform continuity: examples and counterexamples. Heine-Cantor theorem: any continuous function $f : K \subset \mathbb{R} \to \mathbb{R}$ on a compact set $K \subset \mathbb{R}$ is uniformly continuous. Examples of uniformly continuous functions on unbounded sets. Boundedness does not guarantee uniform continuity: the function $f(x) = \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$ despite being both continuous and bounded. A couple of theoretical exercises on continuous functions: prove that any polynomial $p(x) = a_0 + a_1 x + \dots a_n x^n$ with $n = 2k + 1$ must have at least one real root; prove that any continuous function $f : I \to I$, with $I = [0, 1]$, has at least one fixed point x_0 (i.e. $f(x_0) = x_0$).

08.11.19 Binomial coefficients $\binom{n}{k}$, $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots, n$, and their main properties. For any $n \in \mathbb{N}$, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, with $k = 1, 2, \dots, n$. Newton's binomial formula: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. In particular, we have $\sum_{k=0}^{n} \binom{n}{k} = 2^n$: combinatorial interpretation of $\binom{n}{k}$ as the number of subsets of $\{1, 2, \dots, n\}$ with k elements. Various exercises on uniform continuity.

11.11.19 Derivative $f'(x_0)$ of a function f at a point x_0: examples and counterexamples. The derivative $f'(x_0)$ as the slope of the tangent line to the graph of a function at the point $(x_0, f(x_0))$. Left and right derivatives, $f'_-(x_0)$ and $f'_+(x_0)$ respectively. A function is differentiable at x_0 if and only if both $f'_-(x_0)$ and $f'_+(x_0)$ exist as finite real numbers and $f'_-(x_0) = f'_+(x_0)$. Differentiability at a point implies continuity. However, continuity alone is no guarantee for differentiability. The derivative f' as a function obtained out of a function f differentiable at every point of a given (open)interval (a, b). First computations of derivatives: the derivative of a constant function is zero; $Dx^n = nx^{n-1}$, $n \in \mathbb{N}$; $D(\frac{1}{x}) = -\frac{1}{x^2}$; $D(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ if $x > 0$; $D(a^x) = \ln(a) a^x$, for any $a > 0$; $D(e^x) = e^x$; $D(\sin x) = \cos x$; $D(\cos x) = -\sin x$.

13.11.19 $D(\log_a x) = \frac{1}{\ln a \ln x}$ and $D(\ln x) = \frac{1}{x}$, with $x > 0$. Differentiation rules: given any two differentiable functions $f, g : I \subset \mathbb{R} \to \mathbb{R}$, then

- cf is differentiable for any $c \in \mathbb{R}$ and $D(cf) = cD(f)$
- the sum $f + g$ is differentiable and $D(f + g) = D(f) + D(g)$
- the product fg is differentiable and $D(fg) = D(f)g + fD(g)$ (Leibniz's rule)
- the quotient $\frac{f}{g}$ is differentiable at those points $x \in I$ where $g(x) \neq 0$ and $D\left(\frac{f}{g}\right) = \frac{D(f)g - fD(g)}{g^2}$

Chain rule: for any pair of differentiable functions $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ with $f(I) \subset J$, the composition $g \circ f : I \subset \mathbb{R} \to \mathbb{R}$ is still differentiable and $g(f(x))' = g'(f(x))f'(x)$. Derivative of the inverse: if $f : I \subset \mathbb{R} \to \mathbb{R}$ is an invertible differentiable function, then $f^{-1} : f(I) \subset \mathbb{R} \to \mathbb{R}$ is differentiable as well and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ for every $x \in f(I)$ such that $f'(f^{-1}(x)) \neq 0$. Applications: $D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ and $D(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$.

14.11.19 More derivatives: $D(x^\alpha) = \alpha x^{\alpha-1}$ for any $\alpha \in \mathbb{R}$ (with $x > 0$); $D(\arctan x) = \frac{1}{1+x^2}$. Local maxima and minima. Critical points. Fermat's theorem: if a differentiable function $f : (a, b) \to \mathbb{R}$ has a local extreme (either a minimum or a maximum) at $x_0 \in (a, b)$ then x_0 is a critical point, i.e. $f'(x_0) = 0$. Exercises.

15.11.19 Rolle's theorem: if $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there exists at least one interior point $c \in (a, b)$ such that $f'(c) = 0$. Mean value theorem: if $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one interior point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Some consequences. If $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for every $x \in (a, b)$, then f

is constant on (a, b). Monotonicity read in terms of the sign of the derivative: a differentiable function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is monotone non-decreasing (increasing) on (a, b) if and only if $f'(x) \geq 0$ ($f'(x) \leq 0$) for every $x \in (a, b)$; if $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing. The converse, however, is not true: $f(x) = x^3$ is strictly increasing and yet $f'(0) = 0$. Exercises.

18.11.19 More applications of Lagrange's mean value theorem: if f is differentiable on the set $(x_0 - \delta, x_o + \delta) \setminus \{x_0\}$ and $\lim_{x\to x_0} f'(x)$ exists and it is finite, then f is differentiable at x_0 as well, and $f'(x_0) = \lim_{x\to x_0} f'(x)$; if $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is differentiable on (a, b) and its derivative is bounded. i.e. $|f'(x)| \leq L$ for any $x \in (a, b)$, then f is Lipschitz continuous on $[a, b]$, i.e. $|f(x) - f(y)| \leq L|x - y|$, for every $x, y \in [a, b]$. The function $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$, although it is uniformly continuous. In fact, its derivative explodes at 0. Cauchy's mean value theorem: if $f, g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for every $x \in (a, b)$, then there exists at least one interior point $c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$. L'Hôpital's rule.

20.11.19 Points of non-differentiability: corner points, cusps, and points with a vertical tangent. Recap exercises.

21.11.19 Derivative and speed. Convex and concave differentiable functions. f is convex on I iff f' is increasing on I iff f'' is non-negative on I . Non differentiable convex and concave functions.

22.11.19 Inflection points. C^k functions. Convexity and concavity for elementary functions.

25.11.19 Recap exercises: qualitative study of the graph of the functions e^{-x^2} and $e^{-|x|}$ including convexity and inflection points. Hyperbolic functions: $\cosh x$, $\sinh x$, $\tanh x$ and the corresponding inverse functions.

27.11.19 Taylor polynomials. Taylor formula with Peano's remainder (without proof).

28.11.19 Recapitulation of Landau symbols. Proof of Taylor formula with Peano's remainder. Taylor formula with Lagrange's remainder (without proof). Taylor formula for e^x at $x_0 = 0$. Approximation of the Euler's number (Napier's constant) e using the Taylor formula and estimates on the corresponding Lagrange remainder.

29.11.19 Taylor expansions for $\log x$, $\log(1 + x)$, $\sin x$, $\cos x$ and $(1 + x)^\alpha$, $\alpha \in \mathbb{R}$.

02.12.19 Operations on Taylor expansions. Uniqueness of the Taylor expansions. Taylor expansion for the product and for the quotient of two functions. Taylor expansion for the composition two functions and for the inverse function. The use of the Taylor expansion for the computation of limits.

04.12.19 Local behavior of a map: Taylor expansion and characterization of local maxima, local minima and inflection points. Recap exircises on the use of the Tatylor expansion for the computation of limits.

05.12.19 Recap exercises. Introduction to complex numbers. Definition of complex numbers and corresponding field properties.

06.12.19 Absolute value of a complex number. Complex conjugate and inverse of a complex number. Representation of complex numbers in the Cartesian plane. Trigonometric form of complex numbers. Argument of a complex number. Euler formula and exponential form of a complex number.

09.12.19 De Moivre's formula. n-th roots of unity. Exponential of a complex number. Solutions of quadratic equations with complex coefficients. Fundantal Theorem of Algebra.

11.12.19 n-th roots of a complex number. Recap exercises.

12.12.19 Integral calculus. Primitive (anti-derivative) of a function defined on an interval. Absence of jump discontinuity is a necessary condition for the existence of a primitive. A primitive is defined up to a real constant, hence the set of primitive is indexed by a real parameter. Indefinite integral. Primitives of elementary functions. Cauchy problem: choice of a primitive satisfying an initial condition. Examples of primitives: continuity, initial condition and geometrical meaning.

16.12.19 Linearity of the indefinite integral. Integration by parts. Examples and exercises.

18.12.19 Integration by substitutions. Examples and exercises.

19.12.19 Integration of rational functions.

20.12.19 Integration of rational functions (continuation). Definite integrals: introduction and main results. Step functions and their integrals. Upper and lower integrals. Riemann integrable functions and definition of the Riemann integral. Integrability of continuous and piecewise-continuous functions (without proof).

08.01.20 Proof of the integrability of continuous functions. Basic properties of the definite integral: additivity, linearity, positivity and monotonicity.

09.01.20 Integral mean value and the integral mean value theorem. Integral functions. The fundamental theorem of calculus and some of its consequences. Relation between definite and indefinite integrals. Computation of definite integrals.

10.01.20 Integration by parts and by substitution for definite integrals. Computation of areas. Improper integrals. Improper integrals on unbounded domains of integration and improper integrals with unbounded integrands. Convergence and divergence of the integrals $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ and $\int_0^1 \frac{1}{x^\alpha}\,dx$, $\alpha \in \mathbb{R}$.

13.01.20 Properties of improper integrals: additivity, linearity, positivity. Comparison test, absolute convergence test and asymptotic comparison test for improper integrals. General improper integrals on unbounded domains of integrations with unbounded integrands.

16.01.20. Recap. exercises on improper integrals. Introduction to numerical series. General term and sequence of partial sums. Convergent, divergent and indeterminate series. Geometric series. Non-negative (positive) term series. Comparison test and asymptotic comparison test. Stirling formula for $n!$. Integral test for non-negative (positive) term series. Harmonic series. Series of type $\sum_{k=2}^{+\infty} \frac{1}{k^\alpha \log^\beta(k)}$, $\alpha, \beta \in \mathbb{R}$. Alternating series and Leibniz's test. Recap. exercises on numerical series.

17.01.20. Introduction to ordinary differential equations. First order differential equations. General integral and particular integral. Cauchy problem (initial value problem) for first order differential equations. Differential equations with separable variables and their solutions. Recap. exercises.